Uniform Convergence of Fourier Series : A Study

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ABSTRACT
The purpose of this paper is to explore the basic question of the uniform convergence of Fourier series. In this, we study the convergence of Fourier series. Specifically, we give some results about uniform convergence of the Fourier series. It is an explanation on Fourier series that converge uniformly to functions. This paper will not go through the deeper questions of convergence, but requires only the basic principles of introductory real analysis.

Keywords: Fourier series, Convergence, Uniform convergence, Analysis.

INTRODUCTION

In the mathematical field of analysis, uniform convergence is a mode of convergence of functions stronger than pointwise convergence.
A sequence of functions $\{h_n\}$ is said to converge uniformy on an interval [c,d] to a function h if for any $\varepsilon > 0$ and for all p ∈ [c,d] there exists an integer N(independent of p but dependent on ε) such that for all p ∈ [c,d] ,

$$|h_n(p) - h(p)| < \varepsilon, \text{ for all } n \geq N$$

It is clear that every uniformly convergent sequence is pointwise convergent sequence, and the uniform limit function is same as the pointwise limit function.
A Fourier series is an expansion of a periodic function h(p) in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions.
If the numbers c_0 , c_1 , …, c_n, …, d_1 , d_2 , …, d_n , … are derived from a function h by means of Euler – Fourier formulae :

$$c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(p) \cos np \, dp \text{ , n = 0, 1, 2, …} \quad \text{…..(1)}$$

$$d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(p) \sin np \, dp \text{ , n = 0, 1, 2, …} \quad \text{…..(2)}$$

then the series,

$$\frac{1}{2} c_0 = \sum_{n=1}^{\infty} (c_n \cos np + d_n \sin np) \quad \text{…..(3)}$$

is called the Fourier series of h or the Fourier series generated by h, and the coefficients c_n , d_n defined by (1) and (2) as the Fourier coefficients of h.
It is to be noted that the Fourier coefficients have been obtained purely on the assumption that the function f is bounded and integrable on $[-\pi, \pi]$.
There is nothing to suggest that the Fourier series (3) is convergent. In fact the series may not converge at all, and even if it converges, the sum may not be h, though it often is and there is some justification for the hope that the series may converge and have h for its sum. In case the Fourier series of h converges uniformly, the definitons of Fourier constants suggest that its sum will be h, and that h is capable of a unique Fourier series expansion.
Now, we'll state and prove some theorms about the convergence of Fourier series. Firstly, we'll discuss about the Riemann-Lebesgue theorm.

RIEMANN – LEBESGUE THEORM : If a function φ is bounded and integrable on the interval [a, b] , then as n→ ∞ ,

$$C_n = \int_c^d \varphi \cos np \, dp \to 0 \text{ , and } D_n = \int_c^d \varphi \sin np \, dp \to 0 \text{ .}$$

Proof : If a and b both belong to the sae interval of the form $-m\pi \leq p \leq m\pi$ (m be any positive integer), we define h(p) = $\varphi(p)$ in $c \leq p \leq d$, and h(p) = 0 at the remaining points of $[-m\pi\, , m\pi]$; for other real p, h is defined so as to be periodic with period 2π. Then

$$C_n = \int_c^d \varphi \cos np \, dp \to 0 = \int_{-\pi}^{\pi} h \cos np \, dp = \pi c_n \text{ ,}$$

and similarly$D_n = \pi d_n$, where c_n and d_n denote the fouriercofficients of h.
Now, as we know if h is bounded and integrable on $[-\pi\, , \pi]$ and if c_n , d_n are its Fourier coefficients, then

$$\sum_{n=1}^{\infty} (c_n^2 + d_n^2) \text{ converges.}$$

Therefore, C_n , D_n tend to zero.
Again, if c and d do not lie in the same interval of the type $[-m\pi\, , m\pi]$, we can split up the interval [c , d] into a finite number of sub-intervals, each of which lies in an interval of the form $[-m\pi\, , m\pi]$. C_n and D_n can be expressed then as a sum of a (fixed) finite number of terms each of which tends to 0, as n → ∞. Thus C_n and D_n tend to 0.

Now, we'll discuss about The Riemann-Lebesgue Lemma.
A simple application of the Weierstrass M-test will show that if

$$\sum_{n=1}^{\infty} |c_n + d_n| \qquad .....(4)$$

converges, then Equation (3) will converge uniformly and absolutely. As it is a necessary condition for a convergent series that the terms tend to zero as n tends to infinity, we can reasonably conclude that Equations (1) and (2) tend to zero as n tends to infinity in any situation where (4) holds. This observation turns out to be a lot more general than just the case where (4) holds. In fact if f(x) is absolutely integrable then (1) and (2) will tend to zero as n tends to infinity regardless of the limits of integration chosen. This fact is referred to as Riemann-Lebesgue lemma.

THEORM 1 :(The Riemann-Lebesgue Lemma) **:** For any function h(p) absolutely integrable on an interval [c,d],

$$\lim_{n\to\infty} \int_c^d f(p) \cos np \, dp \to 0 = \lim_{n\to\infty} \int_c^d f(p) \sin np \, dp \to 0 = 0$$

Proof : We'll prove the theorem for just the sin np case as the proof for cos np is essentially the same. We begin by first proving the theorem for proper integrals.
We know that,

$$\lim_{n\to\infty} \int_{t_{i-1}}^{t_i} \sin np \, dp = \lim_{n\to\infty} \frac{\cos nt_{i-1} - \cos t_i}{n} = 0$$

We create a partition of the interval [c,d]
c < t_1 < ... < t_{n-1} < d and define a step function s(p) = $\inf_{y\in[t_{i-1},t_i]}$ h(y) for all $p \in (t_{i-1}, t_i)$ leaving s(p) undefined for all t_i. Thus for any partition of [c,d]

$$\lim_{n\to\infty} \int_c^d s(p) \sin np \, dp = 0$$

For all $\epsilon > 0$, there exists some partition of [c,d] that gives

$$\int_c^d |h(p) - s(p)| \, dp = \int_c^d h(p) - s(p) \, dp < \frac{\epsilon}{2}$$

Hence through the triangle inequality, we find

$$| \int_c^d h(p) \sin np \, dp | \leq \int_c^d h(p) - s(p) \, dp + | \int_c^d h(p) \sin np \, dp | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for large enough n.
For proper integrals, it can be seen that we only require that h should be integrable (If a proper integral of h exists, then the proper integral of |h| will also exist). In the case of an improper integral on [c, d], we would now explicitly require that h be absolutely integrable. Without loss of generality assume the improper integral in question is

$$\lim_{l\to d}\int_c^l f(p)\sin np\,dp \qquad .....(5)$$

As f
h is absolutely integrable, for any $\epsilon > 0$ we can choose some t ∈ [c, d] which gives

$$\lim_{l\to d}\int_t^l |f(p)|dp < \frac{\epsilon}{2} \qquad .....(6)$$

The triangle inequality and a little work will then show that the absolute value of (5) is less than or equal to the absolute value of a proper integral of h(p) $\sin np$ on [c, t] and the improper integral shown in the inequality (6). Thus through our proof of the Riemann-Lebesgue lemma for proper integrals and (6), we have that for any $\epsilon > 0$

$$|\lim_{l\to d} \int_c^d h(p)\sin np\,dp\,| < \epsilon$$

for large enough n, completing the proof.
With regard to Fourier Series, the Riemann-Lebesgue lemma tells us that the terms of the series approach zero as n gets larger. Thus the Fourier series of any absolutely integrable function already satisfies a necessary (but not sufficient) condition for point-wise and uniform convergence.
Now, we will state two lemmas which will be directly applied to proving our main theorems of uniform convergence.
Lemma 1 : Let h(p) be an absolutely integrable, periodic function and let $\omega(u)$ be a function with a continuous derivative on [c, d]. Then for any $\epsilon > 0$, the inequality

$$|\int_c^d f(p+u)\omega(u)\sin mu\,du| < \epsilon$$

holds for all p provided m is large enough.

Lemma 2 : The integral

$$I(u) = \int_0^u \frac{\sin mt}{2\sin\frac{t}{2}}dt$$

is bounded on $[-\pi, \pi]$.

Now, as we know our first condition for the uniform convergence of the Fourier series S(h)(p) to the function h(p) would necessarily be that h should be continuous on the interval in question. As the Dirichlet conditions for a continuous point p hint at the role that the derivative of the function may have to play in the convergence of S(h)(p) to h(p). The nature of the derivative of a function does indeed play an important part in ensuring uniform convergence of the function's Fourier series. Now, we'll show that if the derivative of an absolutely integrable periodic h is absolutely integrable over the interval, we obtain uniform convergence.
Consider the partial sum of Fourier series,

$$S_k(h)(p) = \frac{1}{2}c_0 + \sum_{n=1}^{k}(c_n\cos np + d_n\sin np) \qquad .....(7)$$

Substituting Equations (1) and (2) into Equation (7) and using

$$\frac{1}{2} + \cos p + \cos 2p + \ldots + \cos np = \frac{\sin(n+\frac{1}{2})p}{2\sin^{p}/_{2}} \qquad .....(8)$$

which can be derived from applying the trigonometric factor formulae to the product of the denominator on the right-hand side and the sums on the left-hand side, the partial sums of the Fourier series represented by Equation (7) can be reformulated as

$$S_k(h)(p) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(p+u)\frac{\sin(n+\frac{1}{2})u}{2\sin^{u}/_{2}}du \qquad .....(9)$$

Also notice from integrating the left-side of Equation (8) that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})u}{2\sin^{u}/_{2}}du = 1 \qquad .....(10)$$

THEORM 2 : If h(p) is continuous and periodic and h'(p) is absolutely integrable, then $S_n(h)(p)$ converges uniformly to h(p) for all p.
Proof : Using the equations (9) and (10),

$$|S_k(h)(p) - f(p)| = \frac{1}{\pi} | \int_{-\pi}^{\pi} [f(p+u) - f(p)] \frac{\sin(n+\frac{1}{2})u}{2\sin {}^{u}/_{2}} du |$$

We now split the integral on the right-hand side into three integrals sub-intervals $[-\pi, -\delta]$, $[-\delta, \delta]$ and $[\delta, \pi]$ where $0 < \delta < \pi$. Therefore, we'll have that $|S_k(h)(p) - f(p)|$ is less than the sum of the absolute values of these three integral by the triangle inequality.

Now, as we know that $\frac{1}{2\sin {}^{u}/_{2}}$ has a continuous derivative on $[-\pi, -\delta]$ and $[\delta, \pi]$ therefore, by Lemma 1, we can make the absolute value of two of our three integrals as small as we please by choosing k large enough, regardless of p.

For the remaining integral on $[-\delta, \delta]$, integrating by parts gives,

$$| \left([h(p+u) - h(p)] \int_0^u \frac{\sin mt}{2\sin {}^{t}/_{2}} dt) \right| \ {}_{-\delta}^{\delta} - \int_{-\delta}^{\delta} h'(p+u) \int_0^u \frac{\sin mt}{\sin \frac{t}{2}} dt\, du |$$

By Lemma 2, there exists K ≥ $|I(u)|$ on $[-\pi, \pi]$, thus if we also note that $I(u)$ is an even function, then the above will be less than

$$K(|h(p+\delta) + h(p-\delta) = 2h(p)| + \int_{-\delta}^{\delta} |h'(p+u)|\, du)$$

Since h(p) is continuous and h'(p) is absolutely integrable, the above expression can be made as small as possible by choosing δ small enough. Putting this together means that we can make $|S_k(h)(p) - h(p)|$ as small as possible by choosing n (which has no dependency on p) large enough, as claimed.

The function f in the above theorm is continuous on the entire real line. Now, our aim is to prove uniform convergence to a function which is continuous on some arbitrary interval [c, d]. To do this we first make an observation that is important to the study of Fourier series. We'll state a Lemma that will help us get to our final theorm of uniform convergence of Fourier series.

Lemma 3 :(The Riemann Localization Principle) **:** If h(p) and q(p) are two absolutely integrable functions with the same period that are equal on some interval [c, d] then

$S_n(h)(p) - S_n(q)(p)$

converges uniformly to zero for all p ∈ (c, d).

Lemma 4 : If a periodic, absolutely integrable function h(p) is continuous on [c, d] with an absolutely integrable derivative, then the Fourier series S(h)(p) converges uniformly to h(p) on (c, d).

Now, as we've proved various theorms and stated various lemma's for the convergence of Fourier series, we'll now discuss about how quickly the Fourier series of a function can converge uniformly.

Firstly, we notice that if a continuous periodic function h has an absolutely integrable derivative h' , then the Fourier series corresponding to h' is given by

$$S(h')(p) = \sum_{k=1}^{\infty} k[d_k \cos kp - c_k \sin kp]$$

We obtained this result by integrating

$$c'_k = \frac{1}{\pi} \int_{-\pi}^{\pi} h'(p) \cos kp\, dp$$

$$d'_k = \frac{1}{\pi} \int_{-\pi}^{\pi} h'(p) \sin kp\, dp$$

by parts.

Therefore, we find $c'_k = kd_k$ and $d'_k = -kc_k$. If the notation $c_k^{(m)}$ and $d_k^{(m)}$ to represent the Fourier coefficients of the absolutely integrable m^{th} derivative of a function is allowed, we can easily check that $|c_k^{(m)}| = |k^m c_k|$ for even values of m, $|k^m d_k|$ for odd values of m. And $|d_k^{(m)}| = |k^m d_k|$ for even values of m and $|k^m c_k|$ for odd values of m, provided that m – 1 preceding derivatives are also continuous. From these considerations, we immediately obtain the following result.

THEORM 3 : If a continuous function h(p) has m-1 continuous derivatives and an m^{th} absolutely integrable derivative, then the Fourier coefficients of h(p) satisfy the relation

$$\lim_{k\to\infty} k^m c_k = \lim_{k\to\infty} k^m d_k = 0$$

Proof : As a result of the Riemann-Lebesgue lemma,

$$\lim_{k\to\infty} c_k^{(m)} = \lim_{k\to\infty} d_k^{(m)} = 0$$

Since the (m – 1) preceding derivatives are continuous, therefore

$$\lim_{k\to\infty} k^m c_k = \lim_{k\to\infty} k^m d_k = 0$$

Now, we know that if h(p) is continuous and periodic and h'(p) is absolutely integrable, then S_n(h)(p) converges uniformly to h(p) for all x.

Therefore, the Fourier series of the given function h will converge uniformly to h everywhere. Since this theorm shows that the Fourier coefficients of h converge to 0 faster than k^{-m}, we observe how quickly the partial sums of the Fourier series converge to a function. The larger the value of m for the given Fourier series such that,

$$\lim_{k\to\infty} k^m c_k = \lim_{k\to\infty} k^m d_k = 0$$

the better the partial sums of the Fourier series S_n(h) approximate the function for a given n since the subsequent terms in the series will decay to zero more quickly. Increasing m is thus the main consideration when improving the convergence of a given Fourier series, that is when improving the quality of each approximation S_n(h)(p) of h(p).

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